Electricity and Magnetism, Exam 5, 19/05 2017

Exam drafted by (name first examiner) Maxim S. Pchenitchnikov
Exam reviewed by (name second examiner) Steven Hoekstra

12 questions; **with answers**

This is a multiple-choice exam. Write your name and student number on the answer sheet. Clearly mark the answer of your choice on the answer sheet. Only a single answer is correct for every question. The score might be corrected for guessing. Use of a (graphical) calculator is allowed. You may make use of the formula sheet (provided separately). The same notation is used as in the book and lectures, i.e. a bold-face **A** is a vector, T is a scalar.

Table of correct answers

Question	**Correct answer**	**Test on**	**Level**
1	**C**	Maxwell's correction	easy
2	**B**	Ampère-Maxwell's law	moderate
3	**D**	Maxwell's equations	easy
4	**A**	Waves	moderate
5	**C**	Waves	easy
6	**D**	Maxwell tensor	difficult
7	**A**	Intensity of EM wave	moderate
8	**D**	Properties of EM waves in vacuum	moderate
9	**B**	Properties of EM waves in vacuum	moderate
10	**B**	Properties of EM waves in vacuum	easy
11	**C**	Waves	easy
12	**A**	Conductivity and displacement current	difficult

Question 1. Maxwell introduced the term $\mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}$ into Ampère's law based on

A. His careful experiments with the charging capacitor

B. Hertz's experiments on the electromagnetic waves

C. An attempt to eliminate inconsistency in the four electric/magnetic equations

D. Faraday's experiments with magnetic currents

Answer C: Maxwell introduced his term to eliminate inconsistency in the four electric/magnetic equations (which of course did not guarantee that this term was correct). Maxwell was not experimenting on the charging capacitor (A); Hertz made his experiments long after Maxwell fixed Ampère's law (B), and Faraday's experiments resulted in the $-\frac{\partial \mathbf{B}}{\partial t}$ term (D).

Question 2. A magnetic field in a certain region of empty space has components $B_x = -ay;\ B_y = 0;\ B_z = 0$, where *a* is a constant and *x, y, z* are the coordinates of the particular point where we are evaluating the field. Which component of the electric field is changing in time?

A. E_x

B. E_z

C. E_y

D. None

Answer B: Ampère-Maxwell's law: $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}$ (empty space, so that no currents)

$$\boldsymbol{\nabla}\times\mathbf{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ -ay & 0 & 0 \end{vmatrix} = -\frac{\partial}{\partial y}(-ay)\mathbf{k} + \frac{\partial}{\partial z}(-ay)\mathbf{j} = a\mathbf{k} = \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}$$

i.e. *z*-component of **E**.

Question 3. Which of the Maxwell's equations might be best suited to solve the following practical problem: calculate the magnetic field inside a capacitor which is being charged

A. $\boldsymbol{\nabla}\cdot\mathbf{E} = \frac{1}{\epsilon_0}\rho$

B. $\boldsymbol{\nabla}\cdot\mathbf{B} = 0$

C. $\boldsymbol{\nabla}\times\mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}$

D. $\boldsymbol{\nabla}\times\mathbf{B} = \mu_0\mathbf{J} + \mu_0\epsilon_0\frac{\partial \mathbf{E}}{\partial t}$

Answer D: we used this equation (in its integral form) in the lecture to solve exactly this problem (Griffiths, p.335)

Question 4. A sinusoidal wave of frequency 500 Hz has a speed of 350 m/s. How far apart are two points that differ in phase by $\pi/3$?

A. 0.12 m
B. 0.06 m
C. $2\pi/3$ m
D. 0.7 m

Answer A.
The wavelength $\lambda = v/f = 350/500 = 0.7\ m$
Phase difference between two adjacent peaks $2\pi \rightarrow \lambda$.
Phase difference of $\pi/3 \rightarrow\ 0.7/(2\pi)\cdot\pi/3 =\ 0.12\ m$

Question 5. Which function below does **NOT** represent a wave? (*A* and *b* are constants with the appropriate units)

A. $f(z,t) = A/(b(z-vt)^2+1)$

B. $f(z,t) = Asin(kz)cos(kvt)$

C. $f(z,t) = \frac{A}{b(bz^2+vt)\ \ +1}$

D. $f(z,t) = Ae^{b(z+vt)^2}$

Answer C. For a function to represent a wave, the function must have $z \pm vt$ dependence. In B, a standing wave is given that can be decomposed into the sum of two counterpropagating waves.

Question 6. Below you find the Maxwell stress tensor $\overleftrightarrow{\mathbf{T}}$ for a monochromatic plane wave

$$\tilde{\mathbf{E}}(z,t) = \tilde{E}_0 e^{i(kz-\omega t)}\hat{\mathbf{x}};\ \tilde{\mathbf{B}}(z,t) = \frac{1}{c}\tilde{E}_0 e^{i(kz-\omega t)}\hat{\mathbf{y}}$$

The elements of the Maxwell stress tensor are given as

$$T_{ij}\ \equiv \epsilon_0\left(E_iE_j - \frac{1}{2}\delta_{ij}E^2\right) + \frac{1}{\mu_0}\left(B_iB_j - \frac{1}{2}\delta_{ij}B^2\right), \text{where } i,j = x,y,z \text{ and } \delta_{ij} = \begin{cases}1, & i=j\\ 0, & i \neq j\end{cases}$$

Which answer is correct?

A. $\overleftrightarrow{\mathbf{T}} = -\epsilon_0\tilde{E}_0^2 e^{2i(kz-\omega t)}\begin{pmatrix}1 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}$

B. $\overleftrightarrow{\mathbf{T}} = -\epsilon_0\tilde{E}_0^2 e^{2i(kz-\omega t)}\begin{pmatrix}0 & 0 & 1\\ 0 & 1 & 0\\ 1 & 0 & 0\end{pmatrix}$

C. $\overleftrightarrow{\mathbf{T}} = -\epsilon_0\tilde{E}_0^2 e^{2i(kz-\omega t)}\begin{pmatrix}1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & 1\end{pmatrix}$

D. $\overleftrightarrow{\mathbf{T}} = -\epsilon_0\tilde{E}_0^2 e^{2i(kz-\omega t)}\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ 0 & 0 & 1\end{pmatrix}$

Answer D. E has only an *x* component, and **B** has only a *y* component. So all the "off-diagonal" ($i \neq j$) terms are zero. As for the "diagonal" elements:

$$T_{xx} = \epsilon_0 \left[E_x\ E_x\ - \frac{1}{2}(E_x^2)\right] + \frac{1}{\mu_0}\left[-\frac{1}{2}(B_y^2)\right] = \frac{\epsilon_0}{2}E_x^2 - \frac{1}{2\mu_0}B_y^2 = \frac{1}{2}\left[\epsilon_0 E_x^2 - \frac{1}{\mu_0}\frac{1}{c^2}E_x^2\right] = \frac{1}{2}\left[\epsilon_0 E_x^2 - \frac{\epsilon_0\mu_0}{\mu_0}E_x^2\right] = 0$$

$$T_{yy} = \epsilon_0 \left[-\frac{1}{2}(E_x^2)\right] + \frac{1}{\mu_0}\left[B_y\ B_y\ - \frac{1}{2}(B_y^2)\right] = -\frac{\epsilon_0}{2}E_x^2 + \frac{1}{2\mu_0}B_y^2 = \frac{1}{2}\left[-\epsilon_0 E_x^2 + \frac{1}{\mu_0}\frac{1}{c^2}E_x^2\right] = \frac{1}{2}\left[-\epsilon_0 E_x^2 + \frac{\epsilon_0\mu_0}{\mu_0}E_x^2\right] = 0$$

$$T_{zz} = \epsilon_0 \left[-\frac{1}{2}(E_x^2)\right] + \frac{1}{\mu_0}\left[-\frac{1}{2}(B_y^2)\right] = -\frac{\epsilon_0}{2}E_x^2 - \frac{1}{2\mu_0}B_y^2 = -\frac{1}{2}\left[\epsilon_0 \tilde{E}_0^2 + \frac{1}{\mu_0}\frac{1}{c^2}\tilde{E}_0^2\right] = -\frac{1}{2}\left[\epsilon_0 E_x^2 + \frac{\epsilon_0\mu_0}{\mu_0}E_x^2\right] = -\epsilon_0 E_x^2 = -\epsilon_0 \tilde{E}_0^2 e^{2i(kz-\omega t)}$$

$$\overleftrightarrow{\mathbf{T}} = -\epsilon_0 \tilde{E}_0^2 e^{2i(kz-\omega t)} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

NB: This is Problem 9.13 from Griffiths which was assigned for homework.

Question 7. An airplane flying at a distance of 10 km from a radio transmitter receives a signal of intensity 10 $\mu W/m^2$. What is the amplitude of the electric field at the airplane due to this signal?

A. $\cong 87\ mV/m$

B. $\cong 750\ mV/m$

C. $\cong 180\ mV/m$

D. $\cong 20\ mV/m$

Answer A. Intensity $I \equiv \langle S \rangle = \frac{1}{\mu_0}\langle EB \rangle = \frac{1}{c\mu_0}\langle EE \rangle = \frac{1}{2}c\epsilon_0 E_0^2; E_0\ = \sqrt{\frac{2I}{c\epsilon_0}} = \sqrt{\frac{2\cdot 10^{-5}}{3\cdot 10^8 \cdot 8.85\cdot 10^{-12}}} \cong 0.087\ V/m = 87\ mV/m$

Question 8. Students were asked to describe all properties of the plain monochromatic electromagnetic wave in vacuum. One student wrote

$$\tilde{\mathbf{E}}(\mathbf{r}, t) = \tilde{E}_0 e^{i(\mathbf{k}\cdot\mathbf{r} - \omega t)}\hat{\mathbf{n}};\ \tilde{\mathbf{B}}(\mathbf{r}, t) = \frac{1}{c}\left(\hat{\mathbf{k}} \times \tilde{\mathbf{E}}\right)$$

where $\tilde{\mathbf{E}}(\mathbf{r}, t)$ and $\tilde{\mathbf{B}}(\mathbf{r}, t)$ are electric and magnetic fields, respectively, $\hat{\mathbf{k}}$ is the unit vector toward the propagation direction, and the rest of notations is standard (i.e. ω is angular frequency etc).

What does the student miss?

A. Electromagnetic waves are transverse

B. Electric and magnetic fields oscillate in phase

C. Electromagnetic wave is monochromatic

D. Nothing; all properties are listed

Answer D. A and B are taken care of by the cross-product, C is taken care of by an exponent in the electric field; the scaling between E and B and propagation speed are taken care by 1/c before the cross-product. All these were discussed in the lecture.

Question 9. Amplitudes of the electric and magnetic fields of a plain monochromatic wave in vacuum are given as E_0 = 1 kV/m and B_0 = 1 mT, respectively. Find the energy density *u* of the given electromagnetic wave.

A. $u \cong 8 \cdot 10^{-4} J$
B. This in not a plain monochromatic wave
C. $u \cong 8.85 \cdot 10^{-6} J$
D. $u \cong 796 \cdot 10^{-6} J$

Answer B.
For a plain monochromatic wave, $B_0 = E_0/c$, i.e. for the given electric field, the magnetic field amplitude should be $B_0 = 1000/(3 \cdot 10^8) \neq 0.001\, T$.
Alternatively, if you calculate contributions of electric and magnetic fileds into the total energy, you notice that these two are VERY different while we know that for a plain monochromatic wave in vacuum they must be identical.

Question 10. The magnetic vector of a plane electromagnetic wave is described as follows:
$\mathbf{B}(\mathbf{r}, t) = B_0 cos[(10\, meter^{-1})y + (3 \cdot 10^9\, s^{-1})t]\hat{\mathbf{z}}$
In which direction does this wave propagate?

A. $\hat{\mathbf{y}}$
B. $-\hat{\mathbf{y}}$
C. $\hat{\mathbf{x}}$
D. $-\hat{\mathbf{x}}$

Answer B. A positive sign between the *y* and *t* terms in the argument of the cosine function means that the direction of propagation is −y.

Question 11. Four following functions are solutions of the wave equation:
$f_1(x, t) = 2sin[2x - t]$
$f_2(x, t) = 4sin[x - 0.8t]$
$f_3(x, t) = 4cos[x - t]$
$f_4(x, t) = -5cos[1.25x - t]$
Which of these waves has the largest propagation speed?

A. f_1
B. f_2
C. f_3
D. f_4

Answer C. $v = \frac{\omega}{k}$; $v_1 = 0.5$, $v_2 = 0.8$, $v_3 = 1$, $v_4 = 0.8$

Question 12. Consider a parallel-plate capacitor immersed in sea water and driven by a voltage $V = V_0 cos(2\pi \nu t)$. Sea water has permittivity $\epsilon = 81\epsilon_0$, permeability $\mu = \mu_0$, and resistivity $\rho = 0.23\Omega \cdot m$ at the driving frequency $\nu = 4 \cdot 10^8 Hz$.
What is the ratio of the amplitudes of conduction current to displacement current?

A. $(2\pi\nu\epsilon\rho)^{-1}$

B. $\frac{cos(2\pi\nu t)}{2\pi}$

C. $tan(2\pi\nu t)$

D. $\frac{cos(2\pi\nu t)}{2\pi\nu\epsilon\rho}$

Answer A. Conduction current $J_c = \sigma E = \frac{1}{\rho} E = \frac{1}{\rho}\frac{V}{d} = \frac{1}{\rho d} V_0 cos(2\pi\nu t)$
Displacement current: $J_d = \frac{\partial D}{\partial t} = \frac{\partial}{\partial t}(\epsilon E) = \epsilon \frac{\partial}{\partial t}\left[\frac{V_0 cos(2\pi\nu t)}{d}\right] = \frac{\epsilon}{d} 2\pi\nu V_0 \left(-sin(2\pi\nu t)\right)$
Ratio of their amplitudes: $\frac{A_c}{A_d} = \frac{V_0}{\rho d}\frac{d}{2\pi\nu\epsilon V_0} = \frac{1}{2\pi\nu\epsilon\rho} = (2\pi\nu\epsilon\rho)^{-1}$
Griffiths, Problem 7.40